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ANALYTICAL SOLUTION OF THE NONHOMOGENEOUS KINETIC EQUATION IN THE PROBLEM OF ISOTHERMAL SLIP OF A RAREFIED GAS ALONG A SOLID SPHERICAL SURFACE

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An analytical method of solution of a half-space boundary-value problem for the nonhomogeneous kinetic Boltzmann equation with a collision operator in the form of an ellipsoidal-statistic model in the problem of isothermal slip of a rarefied-gas flow along a solid spherical surface is presented. Within the framework of the considered model, a correction to the coefficient of isothermal slip that is due to the wall curvature is obtained. Comparison with literature data is made.

Introduction. The complete system of boundary conditions in the case of flow of a rarefied gas past an arbitrary solid smooth surface has been obtained in [1] by the moments methods with use of the Bhatnagar–Gross–Krook (BGK) model of the kinetic Boltzmann equation. Later, for the case of a solid spherical surface, this problem was solved by the moments methods for both the linearized Boltzmann equation with a collision operator in the Boltzmann form [2] and the ellipsoidal-statistic (ES) model of the kinetic Boltzmann equation [3, 4]. Using the method of elementary solutions (Case method [6]) Gaidukov and Popov [5] obtained an exact closed analytical solution for the coefficient β_R which makes it possible to take into account the dependence of the coefficient of thermal creep on the radius of curvature of the solid spherical surface in flow of a rarefied gas. The value of the coefficient β_R found in [5] by numerical methods is in good agreement with the results obtained earlier in [2, 7, 8].

The present work is aimed at calculating the correction $K_{\text{sl}}^{(1)}$ to the coefficient of isothermal slip that is due to the wall curvature by the method [5] based on the ES model of the kinetic Boltzmann equation given. In contrast to [5], the integrals which enter into the resultant expression are calculated analytically. Account for such a correction is necessary when the boundary conditions of second order in the Knudsen number ($\text{Kn} = \lambda/R$) are used.

Problem Formulation. Derivation of Basic Equations. Let us consider a solid spherical surface in flow of a rarefied gas with small deviations from the state of equilibrium. We tie the spherical coordinate system, the polar axis of which is directed along the mass velocity of the gas at a distance from the surface (see Fig. 1), to the center of curvature of the surface. Then, by virtue of the axial symmetry of the problem, the tangent of the component of the mass velocity U_θ is nonzero.

Due to the presence of the forces of viscous friction in the gas, the quantity U_θ is not constant but slowly changes along the normal to the surface. Thus, the quantity $\partial U_\theta/\partial r$ is nonzero in the problem. Let $U_\theta|_S$ be the velocity of the gas on the surface. Then, at small gradients of mass velocity [6, 9]

$$U_\theta|_S = K_{\text{sl}}\lambda k\,, \quad k = (\partial U_\theta/\partial r)|_S\,.$$

We describe the gas flow by the linearized Boltzmann equation with a collision operator in the form of the ES model [10, 11], which in the spherical coordinate system is written as

$$C_r \frac{\partial f}{\partial r} + \frac{1}{R}\left[C_\theta \frac{\partial f}{\partial \theta} + \frac{C_\varphi}{\sin\theta}\frac{\partial f}{\partial \varphi} + (C_\theta^2 + C_\varphi^2)\frac{\partial f}{\partial C_r} + (C_\varphi^2 \operatorname{ctan}\theta - C_r C_\theta)\frac{\partial f}{\partial C_\theta} - (C_\varphi C_\theta \operatorname{ctan}\theta + C_r C_\varphi)\frac{\partial f}{\partial C_\varphi}\right] =$$

$$= f^0(\mathbf{r}, \mathbf{C})\left(1 + \beta^{-3/2}\iiint K(\mathbf{C}, \mathbf{C}')\, f(\mathbf{r}, \mathbf{C}')\, d^3C_i\right) - f(\mathbf{r}, \mathbf{C})\,. \tag{1}$$

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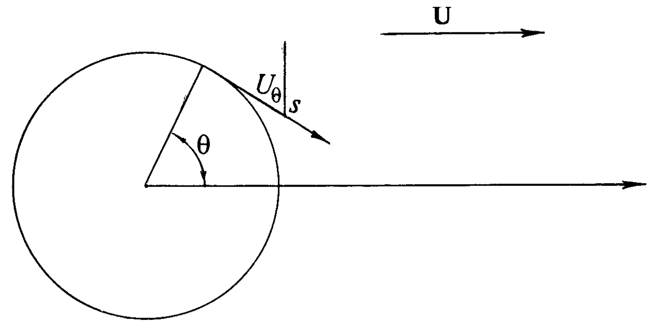

Fig. 1. Problem of flow of a rarefied gas along a spherical surface.

Here $f^0(\mathbf{r}, \mathbf{C}) = (\beta/\pi)^{3/2} \exp(-C^2)$, $\beta = m/2k_BT$, $r(3\mu_g/2p)\,\beta^{-1/2}$ is the dimensional radius vector, and $\beta^{-1/2}U_i$ and $\beta^{-1/2}C_i$ are the components of the mass-mean velocity of the flow and the intrinsic velocity of gas molecules,

$$K(\mathbf{C}, \mathbf{C}') = 1 + \mathbf{C}\mathbf{C}' + \frac{2}{3}\left(C^2 - \frac{3}{2}\right)\left(C'^2 - \frac{3}{2}\right) - 2C_iC_j\left(C_i'C_j' - \frac{1}{3}\delta_{ij}C'^2\right).$$

We take the model of diffusion reflection as the boundary condition on the wall.

Let us linearize $f(\mathbf{r}, \mathbf{C})$ relative to the locally equilibrium distribution function written in the Chapman–Enskog approximation [12]. Expanding the function $Y(r, \theta, C_i)$, which allows for the deviation of the distribution function in the Knudsen layer from the distribution function in the gas volume, into a series in the small parameter $1/R$

$$Y(r, \theta, C_i) = Y^{(1)}(r, \theta, C_i) + R^{-1}Y^{(2)}(r, \theta, C_i) + \ldots \tag{2}$$

we come to the system of one-dimensional integro-differential equations

$$C_r \frac{\partial Y^{(1)}}{\partial r} + Y^{(1)}(r, \theta, C_i) = \pi^{-3/2} \iiint \exp(-C'^2)\, K(\mathbf{C}, \mathbf{C}')\, Y^{(1)}(r, \theta, C_i')\, d^3C_i', \tag{3}$$

$$C_r \frac{\partial Y^{(2)}}{\partial r} + Y^{(2)}(r, \theta, C_i) = \pi^{-3/2} \iiint \exp(-C'^2)\, K(\mathbf{C}, \mathbf{C}')\, Y^{(2)}(r, \theta, C_i')\, d^3C_i' -$$

$$-(C_\theta^2 + C_\varphi^2)\frac{\partial Y^{(1)}}{\partial C_r} - (C_\varphi^2 \operatorname{ctan}\theta - C_rC_\theta)\frac{\partial Y^{(1)}}{\partial C_\theta} + (C_\varphi C_\theta \operatorname{ctan}\theta + C_rC_\varphi)\frac{\partial Y^{(1)}}{\partial C_\varphi} - C_\theta\frac{\partial Y^{(1)}}{\partial \theta} \tag{4}$$

with the boundary conditions

$$Y^{(1)}(R, \theta, C_i) = -2C_\theta U_\theta^{(1)}\Big|_S + \frac{4}{3}C_rC_\theta k, \quad C_r > 0, \quad Y^{(1)}(\infty, \theta, C_i) = 0,$$

$$Y^{(2)}(R, \theta, C_i) = -2C_\theta U_\theta^{(2)}\Big|_S, \quad C_r > 0, \quad Y^{(2)}(\infty, \theta, C_i) = 0,$$

from which we find the form of the first two terms of expansion (2).

Equation (3) describes the processes occurring on the boundary of the solid plane surface and (4) makes it possible to allow for the effect of the curvature of the phase interface.

We find solutions (3) and (4) in the form

$$Y^{(j)}(r, \theta, C_i) = C_\theta Z^{(j)}(r, \theta, C_r), \quad j = 1, 2. \tag{5}$$

Let us introduce the notation $\mu = C_r$. Then, substituting expansions (5) into (3) and (4), multiplying the obtained relation by $C_\theta \exp(-C_\theta^2 - C_\varphi^2)$, and integrating with respect to C_θ and C_φ from $-\infty$ to $+\infty$, we obtain the equation for determining the function $Z^{(1)}(r, \theta, \mu)$ and $Z^{(2)}(r, \theta, \mu)$

$$\mu \frac{\partial Z^{(1)}}{\partial r} + Z^{(1)}(r, \theta, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} Z^{(1)}(r, \theta, \mu') \exp(-\mu'^2)\, d\mu' - \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu' Z^{(1)}(r, \theta, \mu') \exp(-\mu'^2)\, d\mu', \tag{6}$$

$$\mu \frac{\partial Z^{(2)}}{\partial r} + Z^{(2)}(r, \theta, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} Z^{(2)}(r, \theta, \mu') \exp(-\mu'^2)\, d\mu' - \frac{\mu}{\sqrt{\pi}} \int_{-\infty}^{\infty} \mu' Z^{(2)}(r, \theta, \mu') \exp(-\mu'^2)\, d\mu' +$$

$$+ \mu Z^{(1)}(r, \theta, \mu) - 2 \frac{\partial Z^{(1)}}{\partial \mu} \tag{7}$$

with the boundary conditions

$$Z^{(1)}(R, \theta, \mu) = -2U_\theta^{(1)}\Big|_S + \frac{4}{3}\mu k = \frac{4}{3}(\mu + Q_1)\, k, \quad \mu > 0, \tag{8}$$

$$Z^{(1)}(\infty, \theta, \mu) = 0, \tag{9}$$

$$Z^{(2)}(R, \theta, \mu) = -2U_\theta^{(2)}\Big|_S, \quad \mu > 0, \tag{10}$$

$$Z^{(2)}(\infty, \theta, \mu) = 0. \tag{11}$$

In writing (8), we took into account that [11]

$$U_\theta^{(1)}\Big|_S = -\frac{2Q_1}{3} k, \tag{12}$$

The solution of Eq. (6) with boundary conditions (8) and (9) is constructed in [11]:

$$Z^{(1)}(r, \theta, \mu) = \int_0^{\infty} a(\eta, \theta)\, F(\eta, \mu) \exp(-x/\eta)\, d\eta, \quad x = r - R,$$

$$F(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta\, P \frac{1}{\eta - \mu} + \exp(\eta^2)\, \lambda(\eta)\, \delta(\eta - \mu), \quad \lambda(z) = 1 + \frac{1}{\sqrt{\pi}} z \int_{-\infty}^{\infty} \frac{\exp(-\mu^2)}{\mu - z}\, d\mu,$$

$$a(\eta, \theta) = \frac{2k \exp(-\eta^2)\, X(-\eta)}{3\, |\lambda^+(\eta)|^2}, \quad \lambda^{\pm}(\eta) = \lambda(\eta) \pm \sqrt{\pi}\, i\eta \exp(-\eta^2),$$

$$X(z) = \frac{1}{z} \exp\left\{ \frac{1}{\pi} \int_0^{\infty} \frac{\theta(\tau) - \pi}{\tau - z}\, d\tau \right\}, \quad \theta(\tau) - \pi = -\pi/2 - \arctan \frac{\lambda(\tau)}{\sqrt{\pi}\, \tau \exp(-\tau^2)}.$$

Thus, the problem has been reduced to solution of Eq. (7) with boundary conditions (10) and (11).

Account for the Influence of the Surface Curvature on the Coefficient of Isothermal Slip. We find the solution of (7) in the form

$$Z^{(2)}(x, \theta, \mu) = \int_0^{\infty} \psi(\eta, \theta, \mu) \exp(-x/\eta)\, d\eta \,. \tag{13}$$

Substituting (13) into (7) and allowing for the fact that

$$\int_{-\infty}^{\infty} F(\eta, \theta, \mu) \exp(-\mu^2)\, d\mu = 0 \,,$$

where

$$F(\eta, \theta, \mu) = \mu a(\eta, \theta) F(\eta, \mu) - 2a(\eta, \theta) \frac{\partial F}{\partial \mu} \,,$$

we come to the characteristic equation

$$(\eta - \mu)\, \psi(\eta, \theta, \mu) = \frac{1}{\sqrt{\pi}} \eta m(\eta, \theta) + \eta F(\eta, \theta, \mu) \,, \quad m(\eta, \theta) = \int_{-\infty}^{\infty} \psi(\eta, \theta, \mu) \exp(-\mu^2)\, d\mu \,,$$

the general solution of which in the space of generalized functions has the form

$$\psi(\eta, \theta, \mu) = \frac{1}{\sqrt{\pi}} \eta P \frac{1}{\eta - \mu} m(\eta, \theta) + \eta P \frac{1}{\eta - \mu} F(\eta, \theta, \mu) + [m(\eta)\, \lambda(\eta) - \eta a(\eta, \theta)] \exp(\eta^2)\, \delta(\eta - \mu) \,.$$

The solution (13) automatically satisfies boundary condition (11) at infinity. Substituting the found form of the function $\psi(\eta, \theta, \mu)$ into (13) and using boundary condition (10) on the surface, we pass to the singular integral equation with a Cauchy kernel

$$\chi(\mu, \theta) = m(\mu, \theta) \exp(\mu^2)\, \lambda(\mu) + \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{\eta m(\eta, \theta)}{\eta - \mu} d\eta, \quad \mu > 0 \,, \tag{14}$$

where

$$\chi(\mu, \theta) = -2U_{\theta}^{(2)}\Big|_S - \frac{4k}{3}(2\mu^2 + Q_1 \mu - 2) + \mu a(\mu, \theta) \exp(\mu^2) \,. \tag{15}$$

In writing (15), we took into account that [5]

$$\int_0^{\infty} \eta P \frac{1}{\eta - \mu} F(\eta, \theta, \mu) = \frac{4k}{3}(2\mu^2 + Q_1 \mu - 2) \,.$$

Here the symbol Q_n is used to denote the Loyalka integrals [13]:

$$Q_n = \frac{2}{\sqrt{\pi}} \int_0^{\infty} \frac{t^{n+1} \exp(-t^2)\, dt}{X(-t)} \,.$$

Let us introduce the auxiliary function

$$M(z,\theta)=\frac{1}{2\pi i}\int_0^\infty \frac{\eta m(\eta,\theta)}{\eta-z}\,d\eta\,.$$

With account for the form of the limiting values of the function $M(z,\theta)$ on the upper and lower edges of the cut $[0,\infty)$ we reduce (14) to the problem of a jump [14]

$$M^+(\mu,\theta)\,X^+(\mu)-M^-(\mu,\theta)\,X^-(\mu)=\mu\chi(\mu,\theta)\exp(-\mu^2)\frac{X^-(\mu)}{\lambda^-(\mu)}\,,\quad \mu>0\,,$$

which has a solution vanishing at infinity when the condition [11]

$$\int_0^\infty \frac{t\chi(t,\theta)}{X(-t)}\exp(-t^2)\,dt=0\,. \tag{16}$$

is satisfied.

Substituting (15) into (16) and evaluating the integrals which enter into the obtained expression, we obtain

$$U_\theta^{(2)}\Big|_S=-\frac{2k}{3}\,.$$

Hence, with account for (2) and (12) we find the velocity of isothermal slip of the rarefied gas along the solid spherical surface

$$U_\theta\Big|_S=\;U_\theta^{(1)}\Big|_S+\frac{1}{R}\;U_\theta^{(2)}\Big|_S=-\frac{2}{3}\left(Q_1+\frac{1}{R}\right)k\,. \tag{17}$$

Passing in (17) to dimensional quantities and allowing for the fact that for the ES model $R^{-1}=3\mathrm{Kn}/\sqrt{\pi}$, we finally obtain

$$U_\theta\Big|_S=1.146665\,(1-1.665627\ \mathrm{Kn})\,\lambda k\,,$$

whence

$$K_{\mathrm{sl}}=1.146665\,(1-1.665627\ \mathrm{Kn})\,.$$

We write the coefficient of isothermal slip in the form

$$K_{\mathrm{sl}}=K_{\mathrm{sl}}^{(0)}\,(1+K_{\mathrm{sl}}^{(1)}\ \mathrm{Kn})\,.$$

Here $K_{\mathrm{sl}}^{(0)}$ is the coefficient of isothermal slip of the rarefied gas along the solid plane surface and $K_{\mathrm{sl}}^{(1)}$ is the coefficient allowing for the dependence of K_{sl} on the curvature of the phase interface in the approximation which is linear with respect to the Knudsen number.

Thus, $K_{\mathrm{sl}}^{(0)}=-1.665627$.

Conclusions. In the work, we calculated the velocity of slip of a rarefied gas along a solid spherical surface from the solution of the linearized Boltzmann equation with a collision operator in the form of the ES model in the Knudsen layer in the linear approximation with respect to the Kundsen number. In the limiting case ($R\to\infty$), the obtained solution becomes the solution of the Kramers problem [6, 11] (problem of determination of the velocity of isothermal slip of a rarefied gas along a solid plane surface from the kinetic Boltzmann equation). The found exact value of the coefficient $K_{sl}^{(1)}$ is in good agreement with the similar result (–1.721) obtained in [4] with the use of approximate methods.

The results obtained can be used in calculating the rate of thermophoresis of solid spherical aerosol particles.

NOTATION

R, radius of the sphere; S, surface of the sphere; Kn, Knudsen number; λ, mean free path of gas particles; U_θ, tangent to the surface of the component of the dimensionless mass velocity written in the spherical coordinate system; K_{sl}, coefficient of isothermal slip; k, gradient of the mass velocity on the surface; $f(\mathbf{r}, \mathbf{C})$, coordinate- and velocity-distribution function of gas molecules; $f^0(\mathbf{r}, \mathbf{C})$, absolute Maxwellian; $\mathbf{r}$, dimensionless radius vector; $\mathbf{C}$, dimensionless intrinsic velocity of gas molecules; C_r, C_θ, and C_φ, components of the dimensionless intrinsic velocity of molecules written in the spherical coordinate system; r, modulus of the dimensionless radius vector; θ and φ, angular coordinates of the spherical coordinate system; m, mass of gas particles; μ_g, dynamic viscosity of the gas; k_B, Boltzmann constant; p, static pressure of the gas; x, distance reckoned along the normal from the sphere surface; $Y(r, \theta, C_i)$, function allowing for the deviation of the distribution function in the Knudsen layer from the distribution function in the gas volume; μ, radial component of the intrinsic velocity of gas molecules; η, spectral parameter of decomposition; $F(\eta, \mu)$, eigenvectors of the continuous spectrum of the Kramers problem; $\lambda(z)$, Cercignani dispersion function; Px^{-1}, distribution in terms of the principal value in evaluating the integral of x^{-1}; $\delta(x)$, Dirac delta function; $a(\eta, \theta)$, coefficients in the expansion of the solution of the Kramers problem in eigenvectors of the continuous spectrum; i, imaginary unit; $X(z)$, canonic function from the Kramers problem; $\theta(\tau)$, single-valued regular branch of the argument of the function $\lambda^+(\tau)$ fixed at zero by the condition $\theta(0) = 0$.

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